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A UNIVERSAL SURVIVAL RING OF CONTINUOUS FUNCTIONS WHICH IS NOT A UNIVERSAL LYING-OVER RING

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ABSTRACT. The ring R of continuous real-valued functions on the one-point compactification of the discrete space of cardinality $\aleph_1$ is a universal survival ring, yet is not a ULO-ring. Chains of prime ideals of R of cardinality $\mathfrak{c}$ exist. Moreover, R/P is a divided domain for each $P \in \mathrm{Spec}\,(R)$. If the Continuum Hypothesis holds, then there exists a minimal prime ideal P of R such that R/P is an infinite-dimensional valuation domain; however, it is consistent with ZFC that no such minimal primes exist.

1. Introduction. All rings considered below are commutative, with $1 \neq 0$; all ring homomorphisms and ring extensions are unital. If A is a ring, then $Z(A)$ denotes the set of zero-divisors of A; $\mathrm{tq}\,(A) := A_{A\setminus Z(A)}$, the total quotient ring of A; and $\mathrm{Spec}\,(A)$ denotes the set of all prime ideals of A. As usual, "dim(ension)" refers to the Krull dimension. Following [**11**, page 28], we use LO to denote the lying-over property of ring extensions. Recall from [**5**, page 419] that a ring extension $A \subseteq B$ is said to satisfy QLO if, whenever $P \in \mathrm{Spec}\,(A)$ is such that $PB \neq B$; then there exists a $Q \in \mathrm{Spec}\,(B)$ such that $Q \cap A = P$. It is clear that LO $\Rightarrow$ QLO, while any nontrivial ring of fractions (for instance, $\mathbf{Z} \subset \mathbf{Q}$) shows that QLO $\not\Rightarrow$ LO. Slightly modifying terminology from [**11**, page 35], we say that a ring extension $A \subseteq B$ is a *survival extension* if $PB \neq B$ whenever $P \in \mathrm{Spec}\,(A)$. It is clear that each ring extension that satisfies LO must be a survival extension; once again, examples such as $\mathbf{Z} \subset \mathbf{Q}$ show that the converse is false. Note that a survival extension satisfies LO if (and only if) it satisfies QLO.

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