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EQUAL SUMS OF LIKE POWERS AND EQUAL PRODUCTS OF INTEGERS

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ABSTRACT. Several mathematicians have studied the problem of finding two distinct sets of integers $x_1, \dots, x_s$ and $y_1, \dots, y_s$, such that $\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k$, $k = k_1, k_2, \dots, k_n$, where k_i are specified positive integers. The particular case when $k = 1, 2, \dots, n$ is the well-known Tarry-Escott problem. This paper is the first detailed study of the problem of finding two distinct sets of nonzero integers which, in addition to the conditions already mentioned, also satisfy the condition $x_1 x_2 \cdots x_s = y_1 y_2 \cdots y_s$. Parametric or numerical solutions are given in this paper for many diophantine systems of this type, two examples being the system of equations $\sum_{i=1}^{5} x_i^k = \sum_{i=1}^{5} y_i^k$, $k = 1, 2, 3, 5$, and $\prod_{i=1}^{5} x_i = \prod_{i=1}^{5} y_i$, and the system given by the equations $\sum_{i=1}^{8} x_i^k = \sum_{i=1}^{8} y_i^k$, $k = 1, 2, \dots, 6$, and $\prod_{i=1}^{8} x_i = \prod_{i=1}^{8} y_i$. It is also shown that certain diophantine systems with equal sums of powers and equal products do not have any nontrivial solutions. Some open problems are mentioned at the end of the paper.

1. Introduction. The general problem of equal sums of like powers consists in finding two distinct sets of integers x_i, y_i, $i = 1, 2, \dots, s$ such that the sums of the kth powers of the integers in both the sets are equal for several values of k. In other words, the problem is concerned with finding nontrivial solutions of the diophantine system

$$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k, \quad k = k_1, k_2, \dots, k_n, \tag{1.1}$$

where the exponents k_j, $j = 1, 2, \dots, n$ are specified positive integers. When the exponents k_j are taken as the consecutive integers $1, 2, \dots, n$, we get the well-known Tarry-Escott problem of degree n. This paper is concerned with obtaining solutions of diophantine systems of type (1.1)

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