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CONVEXITY AND OSCULATION IN NORMED SPACES

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ABSTRACT. Constructive properties of uniform convexity, strict convexity, near convexity, and metric convexity in real normed linear spaces are considered. Examples show that certain classical theorems, such as the existence of points of osculation, are constructively invalid. The methods used are in accord with principles introduced by Errett Bishop.

Introduction. Contributions to the constructive study of convexity are found in [**1, 2, 9**] and, most extensively, in a recent paper of Fred Richman [**8**]. The present paper consists mainly of comments related to the latter paper, and Brouwerian counterexamples concerning various properties of convexity and the existence of certain points.

Constructive mathematics. A characteristic feature of the constructivist program is meticulous use of the conjunction "or." To prove "P or Q" constructively, it is required that either we prove P, or we prove Q; it is not sufficient to prove the contrapositive $\neg(\neg P$ and $\neg Q)$.

To clarify the methods used here, we give examples of familiar properties of the reals which are constructively *invalid*, and also properties which are constructively *valid*. The following classical properties of a real number α are constructively invalid: "Either $\alpha < 0$ or $\alpha = 0$ or $\alpha > 0$," and "If $\neg(\alpha \leq 0\ )$, then $\alpha > 0$." The relation $\alpha > 0$ is given a strict constructive definition, with far-reaching significance. Then, the relation $\alpha \leq 0$ is defined as $\neg(\alpha > 0)$. A constructively valid property of the reals is the *Constructive Dichotomy lemma*: If $\alpha < \beta$, then for any real number γ, either $\gamma > \alpha$, or $\gamma < \beta$. This lemma is applied ubiquitously, as a constructive substitute for the constructively invalid *Trichotomy*. For more details, see [**1**].

Applying constructivist principles when reworking classical mathematics can have interesting and surprising results.[1]

Brouwerian counterexamples. To determine the specific nonconstructivities in a classical theory, and thereby indicating feasible direc-

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