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ON THE SIGNATURE OF A CLASS OF CONGRUENCE SUBGROUPS

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ABSTRACT. We find explicit formulas for the signatures of a large family of congruence subgroups of $\mathrm{SL}(2,\mathbf{Z})$. The family depends upon five parameters and includes a family of groups first introduced by Larcher. Larcher showed that every (regular) congruence subgroup G contains at least one subgroup H from this family, such that G and H have the same parabolic elements. Thus, every congruence subgroup contains a "large" Larcher subgroup. These facts were used by Sebbar to classify the torsion-free, genus-zero congruence subgroups of $\mathrm{PSL}(2,\mathbf{R})$. The results of this paper have been used by one of the authors to classify the torsion-free, genus-one congruence subgroups of $\mathrm{PSL}(2,\mathbf{R})$.

1. Introduction. Let $\Gamma := \mathrm{SL}(2,\mathbf{Z})$, and define a subgroup H of Γ to be a congruence subgroup if it contains one of the principal congruence subgroups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid (a-1) \equiv (d-1) \equiv b \equiv c \equiv 0 \pmod N \right\}.$$

The smallest N such that $\Gamma(N)$ is contained in H is called the level of H.

Let $\mathfrak{H}$ be the complex upper half-plane and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbf{Q}^*$ where $\mathbf{Q}^* = \mathbf{Q} \cup \{\infty\}$. If H is a subgroup of Γ, then H acts on both $\mathbf{Q}^*$ and $\mathfrak{H}^*$ by fractional linear transformations. If H is a finite index subgroup of Γ, then the number of orbits of H acting on $\mathbf{Q}^*$ is called the cusp number of H. For each $\alpha \in \mathbf{Q}^*$, let H_α be the stabilizer of α in H. The set of cusp widths of H is defined to be $C(H) = \{\mathrm{Index}\,(\overline{\Gamma_\alpha} : \overline{H_\alpha}) \mid \alpha \in \mathbf{Q}^*\}$ where $\overline{\Gamma_\alpha}$ and $\overline{H_\alpha}$ are the images of Γ_α and H_α in $\overline{\Gamma} := \mathrm{PSL}\,(2,\mathbf{Z})$, respectively.

It is a surprising fact that, for any congruence subgroup H, the set of cusp widths $C(H)$ is closed under taking greatest common divisors and

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