ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 1, 2013

GENUS TWO CURVES WITH EVERYWHERE TWISTED GOOD REDUCTION

HOURIA BAAZIZ AND JOHN BOXALL

ABSTRACT. We construct examples of genus two curves C over quadratic fields K with everywhere twisted good reduction, i.e., for any finite prime $\mathfrak{p}$ of K, C has a twist that has good reduction at $\mathfrak{p}$. An analogous construction for elliptic curves enables us to recover Setzer's family of curves with everywhere good reduction over an imaginary quadratic field.

1. Introduction. Let K be a number field, and let X/K be a smooth projective variety. We say that X has good reduction at a finite prime $\mathfrak{p}$ of K if X has a smooth model $\mathcal{X}_{\mathfrak{p}}$ over the local ring at $\mathfrak{p}$. It is well known that X has good reduction outside a finite set $\Sigma(X)$ of primes $\mathfrak{p}$; we say that X has everywhere good reduction if $\Sigma(X)$ is empty. A well-known theorem of Fontaine [**4**] and Abrashkin [**1**] asserts that there are no abelian varieties with everywhere good reduction over $\mathbf{Q}$. On the other hand many authors have given examples of elliptic curves having everywhere good reduction over quadratic fields. By taking products, one obtains abelian varieties of arbitrary dimension with everywhere good reduction over quadratic fields.

When X is a curve of genus at least one, one knows that if X has good reduction at $\mathfrak{p}$ then the jacobian J_X of X also has good reduction at $\mathfrak{p}$. The converse is not true: if, for instance X is of genus two and J_X has good reduction at $\mathfrak{p}$, then the special fiber of $\mathcal{X}_{\mathfrak{p}}$ is either smooth or the union of two curves of genus one intersecting at a point.

This paper grew out of an attempt to find genus two curves over quadratic fields with everywhere good reduction. (Of course, the Fontaine-Abrashkin theorem implies that there are no such curves over $\mathbf{Q}$.) If C is a (smooth projective) genus two curves over a number field K, then C has an affine model of the form $y^2 = P(x)$ where P is a square-free polynomial of degree 5 or 6 with coefficients in K and all

2010 AMS *Mathematics subject classification.* Primary 11G05, 11G30.
Keywords and phrases. Genus two curves, elliptic curves, everywhere good reduction.
Received by the editors on March 22, 2009, and in revised form on July 6, 2010.
DOI:10.1216/RMJ-2013-43-1-55