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ARGUMENTS OF ZEROS OF HIGHLY LOG CONCAVE POLYNOMIALS

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ABSTRACT. For a real polynomial $p = \sum_{i=0}^{n} c_i x^i$ with no negative coefficients and $n \geq 6$, let $\beta(p) = \inf_{i=1}^{n-1} c_i^2/c_{i+1}c_{i-1}$ (so $\beta(p) \geq 1$ entails that p is log concave). If $\beta(p) > 1.45\cdots$, then all roots of p are in the left half plane and, moreover, there is a function $\beta_0(\theta)$ (for $\pi/2 \leq \theta \leq \pi$) such that $\beta \geq \beta_0(\theta)$ entails all roots of p to have arguments in the sector $|\arg z| \geq \theta$ with the smallest possible θ; we determine exactly what this function (and its inverse) is (it turns out to be piecewise smooth, and quite tractible). This is a one-parameter extension of the Hutchinson-Kurtz theorem (which asserts that $\beta \geq 4$ entails all roots are real).

1. Introduction. As an outgrowth of a question concerning a class of analytic functions, we give criteria for all roots of real polynomials to lie in a sector of the form $\{z \in \mathbf{C} \mid |\arg z| > \theta\}$, at least for $\pi \geq \theta \geq \pi/2$ and asymptotically as $\theta \to 0$. The criteria depend only upon log concavity of the coefficients.

Specifically, if $f = \sum_{i=0}^{N} c_i x^i$ (of degree $N \geq 6$) is a polynomial with positive coefficients, let $\beta := \inf_{i=1}^{N-1} c_i^2/c_{i+1}c_{i-1}$ and assume $\beta > 1$. Then there is a $\theta > 0$ such that for all roots, z, of f, $|\arg z| > \theta$ (where arg is the principal value, i.e., arg takes on values in $(-\pi, \pi]$). The function $\beta \mapsto \theta$ is determined exactly for $\pi/2 \leq \theta \leq \pi$. For example, if $\beta = 1 + \sqrt{2}$, then all roots of f lie in the sector $|\arg z| > 3\pi/4$, while, if $\beta = 2$, then all roots lie in $|\arg z| > 2\pi/3$, and moreover, these numbers are sharp.

This is an extension of Kurtz's theorem, which states that if the c_j are all positive and $\beta > 4$, then all roots are real. We provide minor improvements on this result. This result goes back to Hutchinson [**4**].

Then we consider in Section 2 an old question [**7**] and [**1**, Section 4], which was also attacked in [**5**, Theorem 4]. Form the entire function (or

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