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TEST GROUPS FOR WHITEHEAD GROUPS

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ABSTRACT. We consider the question of when the dual of a Whitehead group is a test group for Whitehead groups. This turns out to be equivalent to the question of when the tensor product of two Whitehead groups is Whitehead. We investigate what happens in different models of set theory.

1. Introduction. All groups in this note are abelian. A *Whitehead group*, or *W-group* for short, is defined to be an abelian group A such that $\mathrm{Ext}\,(A, \mathbf{Z}) = 0$. We are looking for groups C other than $\mathbf{Z}$ such that a group A is a W-group if and only if $\mathrm{Ext}\,(A, C) = 0$; such a C will be called a *test group for Whitehead groups*, or a *W-test group* for short. Notice that, if C is a non-zero separable torsion-free group, then $\mathrm{Ext}\,(A, C) = 0$ implies A is a W-group (since $\mathbf{Z}$ is a summand of C), but the converse may not hold, that is, $\mathrm{Ext}\,(A, C)$ may be non-zero for some W-group A.

Among the separable torsion-free groups are the dual groups, where by a dual group we mean one of the form $\mathrm{Hom}\,(B, \mathbf{Z})$ for some group B. We call $\mathrm{Hom}\,(B, \mathbf{Z})$ the ($\mathbf{Z}$-)*dual* of B and denote it by B^*. We shall call a group a *W^*-group* if it is the dual of a W-group. The principal question we will consider is whether every W^*-group is a W-test group. This turns out to be equivalent to a question about tensor products of W-groups. (See Corollary 2.4.)

As is almost always the case with problems related to Whitehead groups, the answer depends upon the chosen model of set theory. We have an easy affirmative answer if every W-group is free (for example in a model of $V = L$); therefore, we will focus on models where there are non-free W-groups. We will exhibit models with differing results about whether W^*-groups are W-test groups, including information about

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