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RINGS WHOSE TOTAL GRAPHS HAVE GENUS AT MOST ONE

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ABSTRACT. Let R be a commutative ring with $\mathrm{Z}(R)$ its set of zero-divisors. In this paper, we study the total graph of R, denoted by $\mathrm{T}(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices and, for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in \mathrm{Z}(R)$. We investigate properties of the total graph of R and determine all isomorphism classes of finite commutative rings whose total graph has genus at most one (i.e., a planar or toroidal graph). In addition, it is shown that, given a positive integer g, there are only finitely many finite rings whose total graph has genus g.

1. Introduction. Let R be a commutative ring with non-zero unity. Let $\mathrm{Z}(R)$ be the set of zero-divisors of R. The concept of the graph of zero divisors of R was first introduced by Beck [**6**], where he was mainly interested in colorings. In his work all elements of the ring were vertices of the graph. This investigation of colorings of a commutative ring was then continued by D.D. Anderson and Naseer in [**2**]. In [**5**], D.F. Anderson and Livingston associate a graph, $\Gamma(R)$, to R with vertices $\mathrm{Z}(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in \mathrm{Z}(R) \setminus \{0\}$, vertices x and y are adjacent if and only if $xy = 0$.

An interesting question was proposed by D.F. Anderson, et al. [**4**]: For which finite commutative rings R is $\Gamma(R)$ planar? A partial answer was given in [**1**], but the question remained open for local rings of order 32. In [**12**] and then independently in [**7, 13**] it is shown that there is no ring of order 32 whose zero-divisor graph is planar.

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