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LINEAR MAPS PRESERVING GENERALIZED INVERTIBILITY ON COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let A and B be unital complex Banach algebras such that B is commutative and semi-simple. We study linear maps from A into B that preserve generalized invertibility.

1. Introduction and preliminaries. Let A be an algebra. An element $a \in A$ is generalized invertible (or regular) if there exists a $b \in A$ such that $aba = a$. We denote by $\mathcal{G}(A)$ the subset of all generalized invertible elements of A. If A is unital, obviously, $A^{-1} \subseteq \mathcal{G}(A)$, where A^{-1} denotes the group of all invertible elements of A. The well known Gleason-Kahane-Zelazko theorem [**11, 15, 24**] states that if A and B are complex unital Banach algebras such that B is commutative and semi-simple, and if $\phi : A \to B$ is a linear map preserving invertibility (i.e., $\phi(a) \in B^{-1}$ whenever $a \in A^{-1}$), then $\phi(1)^{-1}\phi$ is multiplicative. It seems natural to devote some attention to the case where ϕ preserves generalized invertibility instead of invertibility. The research into this area was initiated in the noncommutative case by Mbekhta, Rodman and Šemrl [**20**]. Afterwards, it was developed in several directions (see [**8, 9, 13, 18, 21**]). It should be pointed out that all these studies are closely connected with Kaplansky's conjecture [**17**]. For more details on this topic, the reader is referred to [**3, 10**].

Now let us define the basic concepts of this note. An algebra A is said to be semi-prime if the condition $aAa = \{0\}$ implies that $a = 0$, for all $a \in A$. Obviously, a semi-simple algebra is semi-prime. Let A be a semi-prime algebra. If A contains minimal left ideals, then the sum of all minimal left ideals is called the socle of A and is denoted by $\operatorname{soc} A$. If A does not have minimal left-ideals we define $\operatorname{soc} A = \{0\}$. It

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