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# $\rho$-HOMOGENEOUS BINOMIAL IDEALS AND PATIL BASES

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ABSTRACT. The paper first generalizes the construction of generating sets for binomial ideals as given in [**6**]. For this, $\rho$-homogeneous binomial ideals are introduced. The resulting generating sets are called Patil bases. It is shown that they are reduced and normalized Gröbner bases. An algorithm for binomials is given to obtain a minimal generating set from a Patil basis. This is applied to the particular case of Patil bases of prime ideals $\mathfrak{p}(n_1,\dots,n_4)$ generated by $\{x^\alpha - x^\beta \mid \alpha,\beta\in\mathbf{N}^4,\ (\alpha-\beta)(n_1,\dots,n_4)^T=0\}$ in $K[x_1,\dots,x_4]$, $K$ a field. We note that our ideals are toric ideals, see [**7**].

**0. Introduction and notation.** Assume $K$ is a field and $R := K[x_1,\dots,x_r]$ the polynomial ring in $r$ indeterminates over $K$, $\mathfrak{m} := (x_1,\dots,x_r)R$. Let $T := \{x_1^{\alpha_1}\cdot\dots\cdot x_r^{\alpha_r} =: x^\alpha \mid \alpha := (\alpha_1,\dots,\alpha_r)\in\mathbf{N}^r\}$ ($\mathbf{N}$ is the set of nonnegative integers) be the set of terms in $R$ and $<$ an admissible term order on $T$. (For undefined terminology for Gröbner bases we refer the reader to [**2**]).

*Remark* 1. If clear from the context, we will use interchangeably the symbol $<$ to denote an admissible term order as well on $T$ as on $\mathbf{N}^r$, i.e. $\alpha<\beta$ means the same as $x^\alpha < x^\beta$. $x^\alpha \mid x^\beta$ (or equivalently $\alpha\mid\beta$) denotes monomial division, $x^\alpha \parallel x^\beta$ (or equivalently $\alpha\parallel\beta$) proper division.

**Definition 0.1.** Assume $f\in R$, $f=\sum_{t\in T} a_t\cdot t$ with $a_t\in K$ and $a_t=0$ for almost all $t\in T$. Then $\operatorname{supp}(f) := \{t \mid t\in T,\ a_t\neq 0\}$.

Let $n_1,\dots,n_r$ be positive integers with $\gcd(n_1,\dots,n_r)=1$. Define the weighted degree for $R$ by $\deg x_i := n_i$, $1\le i\le r$. Let

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