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ON THE EXISTENCE OF ZERO-SUM SUBSEQUENCES OF DISTINCT LENGTHS

BENJAMIN GIRARD

ABSTRACT. In this paper, we obtain a characterization of short normal sequences over a finite abelian p-group, thus answering positively a conjecture of Gao for a variety of such groups. Our main result is deduced from a theorem of Alon, Friedland and Kalai, originally proved so as to study the existence of regular subgraphs in almost regular graphs. In the special case of elementary p-groups, Gao's conjecture is solved using Alon's Combinatorial Nullstellensatz. To conclude, we show that, assuming every integer satisfies Property B, this conjecture holds in the case of finite abelian groups of rank two.

1. Introduction. Let $\mathcal{P}$ be the set of prime numbers, and let G be a finite abelian group, written additively. By $\exp(G)$ we denote the exponent of G. If G is cyclic of order n, it will be denoted by C_n. In the general case, we can decompose G as a direct product of cyclic groups $C_{n_1} \oplus \cdots \oplus C_{n_r}$ where $1 < n_1 | \cdots | n_r \in \mathbf{N}$. For each g in G, we denote by $\mathrm{ord}\,(g)$ its order in G, and by $\langle g \rangle$ the subgroup it generates.

By a *sequence* over G of *length* ℓ, we mean a finite sequence of ℓ elements from G, where repetitions are allowed and the order of elements is disregarded. We use multiplicative notation for sequences. Let

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_{g(S)}}$$

be a sequence over G, where, for all $g \in G$, $\mathsf{v}_g(S) \in \mathbf{N}$ is called the *multiplicity* of g in S. We call $\mathrm{Supp}\,(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\}$ the *support* of S and $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} g \mathsf{v}_g(S)$ the *sum* of S. In addition, we say that $s \in G$ is a *subsum* of S when

$$s = \sum_{i \in I} g_i \text{ for some } \varnothing \subsetneq I \subseteq \{1, \ldots, \ell\}.$$

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