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ON GENERALIZATIONS AND REINFORCEMENTS OF A HILBERT'S TYPE INEQUALITY

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ABSTRACT. In this paper, by using the Euler-Maclaurin expansion for the zeta function, we establish an inequality of a weight coefficient. Using this inequality, we derive generalizations and reinforcements of a Hilbert's type inequality.

1. Introduction. If $p, q > 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in \mathbf{N}$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}, \tag{1.2}$$

where the constant $\pi/(\sin(\pi/p))$ and pq is best possible for each inequality, respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [**1**].

In [**4, 6, 7**], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [**2**], Kuang Jichang and Debnath gave a reinforcement of inequality (1.2):

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