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ON THE NUMBERS OF FACES OF LOW-DIMENSIONAL REGULAR TRIANGULATIONS AND SHELLABLE BALLS

CARL W. LEE AND LAURA SCHMIDT

ABSTRACT. We investigate the conjectured sufficiency of a condition for h-vectors $(1, h_1, h_2, \dots, h_d, 0)$ of regular d-dimensional triangulations. (The condition is already shown to be necessary in [**2**]). We first prove that the condition is sufficient when $h_1 \geq h_2 \geq \cdots \geq h_d$. We then derive some new shellings of squeezed spheres and use them to prove that the condition is sufficient when $d = 3$. Finally, in the case $d = 4$, we construct shellable 4-balls with the desired h-vectors, showing them to be realizable as regular triangulations when $h_4 = 0$ or $h_4 = h_1$.

1. Introduction.

1.1. Polytopes and the g-theorem. The g-theorem [**2, 13**] characterizes the f-vectors of simplicial (and hence also simple) convex polytopes. One corollary is a necessary condition for the f-vectors of simple unbounded polyhedra [**2**], which are the duals of regular triangulations [**15**]. In this paper we investigate the sufficiency of this condition, verifying it in several cases.

We begin with some definitions; more details can be found, for example, in [**4, 15**]. A *convex polyhedron* is an intersection of finitely many closed halfspaces in $\mathbf{R}^d$. A bounded convex polyhedron is called a *convex polytope*. The *f-vector* of a d-dimensional polyhedron (d-polyhedron) P is $f(P) = (f_0(P), \dots, f_{d-1}(P))$, where $f_j(P)$ denotes the number of j-faces of P. We also take $f_{-1}(P) = f_d(P) = 1$. A d-polytope is *simplicial* if every face is a simplex, and *simple* if every vertex (0-face) is contained in exactly d edges (1-faces), equivalently, in exactly d facets ($(d-1)$-faces). Simple polytopes are precisely the duals of simplicial polytopes.

The *h-vector* of a simplicial d-polytope P (or of any $(d-1)$-dimensional simplicial complex) is $h(P) = (h_0(P), \dots, h_d(P))$, where

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