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ON ℓ_ψ SPACES AND INFINITE ψ-DIRECT SUMS OF BANACH SPACE

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ABSTRACT. In this paper, the sequence Banach spaces ℓ_ψ are defined for a class of convex functions ψ and properties of these spaces and their dual spaces are proved. It can be seen that some well-known sequence Banach spaces are spaces of this type. The ψ-direct sum of a sequence $(X_n)_{n\in\mathbf{N}}$ of Banach spaces is also defined. The modulus of convexity of this space is estimated in terms of the modulus of convexity of the spaces ℓ_ψ and X_n, $n = 1, 2, \dots$. Based on this estimate, conditions are proved under which uniform convexity, uniform smoothness and uniform non-squareness are inherited by ψ-direct sums.

1. Introduction. A norm $\|\cdot\|$ on $\mathbf{C}^2$ is called absolute if $\|(z_1, z_2)\| = \|(|z_1|, |z_2|)\|$ for $(z_1, z_2) \in \mathbf{C}^2$ and is called normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. For a continuous and convex function ψ on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1 - s, s\} \le \psi(s) \le 1$ for every $0 \le s \le 1$, Bonsall and Duncan [**3**] defined the norm

$$\|(z_1, z_2)\|_\psi = \begin{cases} (|z_1| + |z_2|)\psi(\frac{|z_2|}{|z_1|+|z_2|}) & \text{if } (z_1, z_2) \neq (0, 0) \\ 0 & \text{if } (z_1, z_2) = (0, 0) \end{cases}$$

on $\mathbf{C}^2$, and they proved that a norm $\|\cdot\|$ on $\mathbf{C}^2$ is absolute and normalized if and only if there is a function ψ with the above properties such that $\|\cdot\| = \|\cdot\|_\psi$. ℓ_p norms are typical examples of such norms, but there are plenty non ℓ_p-type norms on $\mathbf{C}^2$ which are absolute and normalized.

For every ψ as above, Takahashi, Kato and Saito [**19**] introduced the ψ-direct sum $X \bigoplus_\psi Y$ of Banach spaces X and Y equipped with

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