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SURJECTIVE LINEAR MAPS PRESERVING CERTAIN SPECTRAL RADII

M. BENDAOUD AND M. SARIH

ABSTRACT. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$. In this paper, we prove that a surjective linear map ϕ from $\mathcal{L}(\mathcal{H})$ into itself preserves the spectral radius $r_1(.)$ if and only if ϕ is an automorphism multiplied by a unimodular scalar. We also consider the case when $\mathcal{H}$ is a finite dimensional Hilbert space and prove that a linear map ϕ from $M_n(\mathbf{C})$ into itself preserves the spectral radius $r_1(.)$ if and only if ϕ is either an automorphism or an anti-automorphism multiplied by a unimodular scalar. Finally, we use this result to show that a linear map ϕ from $M_n(\mathbf{C})$ into itself preserves the inner local spectral radius at nonzero fixed vector $x_0 \in \mathbf{C}^n$ if and only if there exist a unimodular scalar $\alpha \in \mathbf{C}$ and an invertible matrix $A \in M_n(\mathbf{C})$ such that $A(x_0) = x_0$ and $\phi(T) = \alpha ATA^{-1}$ for all $T \in M_n(\mathbf{C})$.

1. Introduction. Let X be a complex Banach space, and let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators on X. For an operator $T \in \mathcal{L}(X)$, we denote the spectrum by $\sigma(T)$ and the approximate point spectrum by $\sigma_{ap}(T)$. We also denote as usual the spectral radius of T by $r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}$ which coincides, by Gelfand's formula for the spectral radius, with the limit of the convergent sequence $(\|T^n\|^{1/n})_n$. The minimum modulus of T is $m(T) := \inf\{\|Tx\| : \|x\| = 1\}$, and is positive precisely when T is injective and has a closed range. Note that the sequence $(m(T^n)^{1/n})_n$ converges and its limit, denoted by $r_1(T)$, coincides with its supremum. In [**12**], Makai and Zemánek proved, in fact, that $r_1(T)$ is nothing but the minimum modulus of $\sigma_{ap}(T)$.

The local resolvent of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$, $\rho_T(x)$, is the set of all $\lambda \in \mathbf{C}$ for which there exists an open neighborhood U_λ

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