ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 2, 2011

ASYMPTOTIC STATISTICAL CHARACTERIZATIONS OF p-HARMONIC FUNCTIONS OF TWO VARIABLES

DAVID HARTENSTINE AND MATTHEW RUDD

Dedicated to Professor Lloyd Jackson and his enduring legacy

ABSTRACT. Generalizing the well known mean-value property of harmonic functions, we prove that a p-harmonic function of two variables satisfies, in a viscosity sense, two asymptotic formulas involving its local statistics. Moreover, we show that these asymptotic formulas characterize p-harmonic functions when $1 < p < \infty$. An example demonstrates that, in general, these formulas do not hold in a non-asymptotic sense.

1. Introduction. A fundamental and fascinating fact about harmonic functions is their characterization by the mean value property [**4**]: the continuous function u is harmonic in the domain $\Omega \subset \mathbf{R}^N$ if and only if

$$(1)\qquad u(x) = ⨍_{\partial B_r(x)} u(s)\, ds = ⨍_{B_r(x)} u(y)\, dy \quad \text{for each } x \in \Omega,$$

where $B_r(x) \Subset \Omega$ is a ball with center x and radius $r > 0$, $\partial B_r(x)$ is its boundary, and $⨍_E f$ denotes the average of f over the set E. Ostensibly, identity (1) says nothing about derivatives and could be studied entirely within the category of continuous functions. It is the prototypical *statistical* characterization of solutions of a PDE, and it is natural to wonder if this is peculiar to Laplace's equation. In other words, can one characterize solutions of other PDEs in a statistical way that avoids any explicit mention of derivatives?

Recent work shows that such statistical characterizations exist, in a certain sense, for p-harmonic functions, i.e., solutions of the quasilinear PDE

$$(2)\qquad -\Delta_p u := -\text{div}\,\left(|Du|^{p-2} Du\right) = 0, \quad \text{for } 1 < p < \infty.$$

2010 AMS *Mathematics subject classification.* Primary 35J92, 35J70, 35D40.
Keywords and phrases. p-Laplacian, mean-value property, p-harmonic functions, viscosity solutions.
Received by the editors on February 2, 2010.

DOI:10.1216/RMJ-2011-41-2-493