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# MULTIDIMENSIONAL GRAPH COMPLETIONS AND CELLINA APPROXIMABLE MULTIFUNCTIONS

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ABSTRACT. Relying on the continuous approximate selection method of Cellina, ideas and techniques from Sobolev spaces can be applied to the theory of multifunctions and differential inclusions. The first part of this paper introduces a concept of *graph completion*, which extends the earlier construction in [**12**] to functions of several space variables. The second part introduces the notion of *Cellina $W^{1,p}$-approximable multifunction.* To show its relevance, we consider the Cauchy problem on the plane $\dot{x} \in F(x)$, $x(0) = 0 \in \mathbf{R}^2$. If $F$ is an upper semicontinuous multifunction with compact but possibly non-convex values, this problem may not have any solution, even if $F$ is Cellina-approximable in the usual sense. However, we prove that a solution exists under the assumption that $F$ is Cellina $W^{1,1}$-approximable.

**1. Introduction.** For a vector-valued function of a scalar variable, the concept of a *graph completion* was introduced in [**12**]. Its main motivation came from control theory. The control of mechanical systems by means of active constraints [**9, 11, 19, 22**] leads to a system of equations of the form

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} f_k(x)\dot{u}_k. \tag{1.1}$$

Here $t \mapsto x(t) \in \mathbf{R}^n$ describes the state of the system, while $t \mapsto u(t) \in \mathbf{R}^m$ is the control function. An upper dot denotes derivative with respect to time. Assume that each $f_k$ is a globally Lipschitz continuous vector field on $\mathbf{R}^n$. Since the right hand side of (1.1) contains the time derivatives $\dot{u}_k$, given an initial data

$$x(0) = \overline{x}, \tag{1.2}$$

to achieve existence and uniqueness of the solution it is natural to consider control functions $t \mapsto u(t) = (u_1, \dots, u_m)(t)$ which are

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