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INTEGRABILITY OF TWO DIMENSIONAL QUASI-HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In terms of the conservative-dissipative decomposition of a vector field, we characterize the two dimensional quasi-homogeneous polynomial differential systems with a polynomial first integral (in these systems, polynomial integrability and analytic integrability are equivalent). We also provide an easy method to allow us to compute them and their centers. Finally, as an application, we find the quasi-homogeneous polynomial systems of degree two which have a polynomial first integral (the cubic homogeneous systems, among others, are included).

1. Introduction. In this paper, we deal with polynomial differential systems

$$(\dot{x}, \dot{y})^T = \mathbf{F}_r = (P, Q)^T, \tag{1}$$

where $\mathbf{F}_r$ is a quasi-homogeneous polynomial vector field of degree $r \in \mathbf{N} \cup \{0\}$ with respect to type $\mathbf{t} = (t_1, t_2) \in \mathbf{N}^2$, i.e., for any arbitrary positive real ε, $P(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^{r+t_1}P(x, y)$, $Q(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^{r+t_2}Q(x, y)$. In the particular case that $\mathbf{t} = (1, 1)$, system (1) is a homogeneous polynomial differential system of degree $r + 1$.

We recall that a function H is a first integral of (1) in an open subset U of $\mathbf{R}^2$ if H is a non-constant function in U which is constant on each solution curve of (1). Clearly, if $H \in \mathcal{C}^1(U)$ verifies $\nabla H \cdot \mathbf{F}_r \equiv 0$. If there exists a polynomial (analytic) first integral of (1), it says that it is polynomially (analytically) integrable.

We are interested in analyzing when system (1) is analytically integrable. For systems (1), it is easy to prove that the analytic integrability

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