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ON POLAR LEGENDRE POLYNOMIALS

H. PIJEIRA CABRERA, J.Y. BELLO CRUZ AND W. URBINA ROMERO

ABSTRACT. In this paper we introduce a new class of polynomials $\{P_n\}$, called polar Legendre polynomials. They appear as solutions of an inverse Gauss problem for the equilibrium position of a field of forces with $n+1$ unit masses. We study algebraic, differential, and asymptotic properties of these polynomials, which are simultaneously orthogonal with respect to a differential operator and a discrete-continuous Sobolev type inner product.

1. Introduction. Let $\{L_n\}_{n\in\mathbf{N}}$ be the monic Legendre polynomials. It is well known that L_n satisfies the following orthogonality relation

$$(1) \qquad \int_{-1}^{1} L_n(x)x^k\,dx = 0, \quad k = 0, 1, \dots, n-1,$$

the second order linear differential equation

$$(2) \qquad -n(n+1)L_n(z) = \big((1-z^2)L_n'(z)\big)',$$

and the so-called *structure relation* [**7**, (4.5.5)]

$$(3) \qquad (z^2-1)L_n'(z) = n\,L_{n+1}(z) - \frac{n^2(n+1)}{4n^2-1}L_{n-1}(z).$$

For a fixed complex number ζ, that in the sequel is called the pole, let us define $P_n = P_{\zeta,n}$ as a monic polynomial such that

$$(4) \qquad (n+1)\,L_n(z) = ((z-\zeta)\,P_n(z))' = P_n(z) + (z-\zeta)P_n'(z).$$

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