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ALTERNATING SUBSETS AND PERMUTATIONS

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ABSTRACT. We give new proofs of theorems on alternating subsets of integers by means of bijective transformations. It is shown that all known results are consequences of a simple result on the residue class of an integer. The notion of alternating subset is extended to permutations of $\{1, 2, \dots, n\}$. In particular, we obtain solutions to the problems of Terquem and Skolem's generalization for permutations.

1. Introduction. A finite, increasing, sequence of natural numbers $(x_1, x_2, \dots)$ is called *alternating* [**5**] if it fulfills the condition

$$x_i \not\equiv x_{i-1} \pmod 2, \quad i > 1. \tag{1}$$

The empty sequence and the 1-term sequence are also alternating sequences by convention.

Such sequences are known as *alternating subsets of integers* (see for example [**1, 4, 10**]). In particular, we recall the fundamental result [**1, 2**]:

The number $h(n, k)$ of alternating k-subsets of $\{1, 2, \dots, n\}$ is given by

$$h(n,k) = \binom{\lfloor \frac{n+k}{2} \rfloor}{k} + \binom{\lfloor \frac{n+k-1}{2} \rfloor}{k}, \tag{2}$$

where $\lfloor N \rfloor$ denotes the greatest integer $\leq N$. It is known that $\sum_{k>0} h(n,k) = F_{n+3} - 2$, where F_N is the Nth Fibonacci number. We will adopt the notation $[n] = \{1, 2, \dots, n\}$.

We consider generalizations of (2) and show that practically all known results are consequences of the following simple lemma on the residue

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