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THE GEOMETRY OF FILIFORM NILPOTENT LIE GROUPS

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ABSTRACT. We study the geometry of a filiform nilpotent Lie group endowed with a left-invariant metric. We describe the connection and curvatures, and we investigate necessary and sufficient conditions for subgroups to be totally geodesic submanifolds. We also classify the one-parameter subgroups which are geodesics.

1. Introduction. Nilpotent Lie groups endowed with left invariant metrics (*nilmanifolds*) arise naturally in many areas of mathematics, including algebra, dynamics and control theory, and, in geometry, they are studied in homogeneous geometry, spectral geometry, subriemannian geometry and harmonic analysis. There has been extensive study of the geometry of two-step nilmanifolds and their compact quotients. In his foundational work [**7, 8**], Eberlein investigates nonsingular two-step nilmanifolds, describing curvatures, geodesics, totally geodesic submanifolds and density of closed geodesics in compact quotients. By now the geometry of two-step nilmanifolds is well understood (see [**9, 10**]), particularly groups with additional structure, such as those of Heisenberg type (see [**4**]). The geometry of two-step nilmanifolds provides the setting for many of the examples of isospectral, nonisometric spaces (see [**6, 12**]).

The geometry of higher-step nilpotent Lie groups is as yet unexplored. Gornet uses three-step nilmanifold geometry to construct examples with some prescribed spectral properties [**13, 14, 15**]. Lauret analyzes preferred ("minimal") metrics on general nilmanifolds [**18, 20, 21**]. Soliton metrics on higher-step nilmanifolds have been studied in low dimensions [**24**] and for several infinite families [**19, 23**]. It is time for

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