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SEMIDEFINITENESS WITHOUT HERMITICITY

CHARLES R. JOHNSON AND ROBERT REAMS

ABSTRACT. Let $A \in M_n(\mathbf{C})$. We provide a rank characterization of the semidefiniteness of Hermitian A in two ways. We show that A is semidefinite if and only if $\operatorname{rank}[X^*AX] = \operatorname{rank}[AX]$, for all $X \in M_n(\mathbf{C})$, and that A is semidefinite if and only if $\operatorname{rank}[X^*AX] = \operatorname{rank}[AXX^*]$, for all $X \in M_n(\mathbf{C})$. We show that, if A has semidefinite Hermitian part and A^2 has positive semidefinite Hermitian part, then A satisfies row and column inclusion. Let $B \in M_n(\mathbf{C})$, and let k be an integer with $k \geq 2$. If $B^*BA, B^*BA^2, \dots, B^*BA^k$ each has positive semidefinite Hermitian part; we show that $\operatorname{rank}[BAX] = \operatorname{rank}[X^*B^*BAX] = \cdots = \operatorname{rank}[X^*B^*BA^{k-1}X]$, for all $X \in M_n(\mathbf{C})$. These results generalize or strengthen facts about real matrices known earlier.

1. Introduction. In [**6**], a number of results about real matrices, not necessarily symmetric, with semidefinite real quadratic form were given. In some cases these results generalize to complex matrices with semidefinite Hermitian part, and, in some, they do not. Here, we sort out what happens in the complex case, and in some instances give new or stronger results. If the proofs in the complex case extend naturally from the real case, by merely changing "transpose" to "transpose complex conjugate," we skip the proof and only refer to [**6**]. However, we have allowed some overlap of material, for the purpose of clarity.

Let $A \in M_n(\mathbf{C})$. The Hermitian part of A is defined in [**3**] by $H(A) = (A + A^*)/2$. A matrix $A \in M_n(\mathbf{C})$ is called positive semidefinite if it is Hermitian ($A^* = A$) and $x^*Ax \geq 0$ for all $x \in \mathbf{C}^n$. $A \in M_n(\mathbf{C})$ is said to have positive semidefinite Hermitian part if $H(A)$ is positive semidefinite. We say $A \in M_n(\mathbf{C})$ is semidefinite if either A or $-A$ is positive semidefinite.

The principal submatrix of A lying in rows and columns $\alpha \subseteq \{1, \dots, n\}$ will be denoted by $A[\alpha]$, and the submatrix lying in rows α

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