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FOUR-DIMENSIONAL LIE ALGEBRAS WITH A PARA-HYPERCOMPLEX STRUCTURE

NOVICA BLAŽIĆ AND SRDJAN VUKMIROVIĆ

Novica Blažić passed away on Monday, 10 October 2005
and this paper is dedicated to his memory

ABSTRACT. In this paper we classify four-dimensional real Lie algebras $\mathfrak{g}$ admitting an integrable, left invariant, para-hypercomplex structure. The equivalence classes of compatible structures are classified. The metric of split signature (2; 2), canonically determined by the para-hypercomplex structure, is very convenient in understanding the structure of $\mathfrak{g}$. Moreover, these structures provide many examples of left invariant metrics of anti-self-dual metric of split signature. Conformal geometry and the curvature of the canonical metric on the corresponding Lie groups are also discussed. For example, the holonomy algebras of this canonical metrics are determined.

1. Introduction. Invariant structures of complex or quaternionic type on Lie groups are important from a geometric view point as well as from algebraic view point. For example, Snow [**11**] and Ovando [**10**] classified the invariant complex structures on four-dimensional, solvable, simply-connected real Lie groups. Invariant hypercomplex structures on four-dimensional real Lie groups are classified by Barberis [**5**] (see Section 2 for details). There is the unique (up to a homothety) positive definite Hermitan metric associated with such a structure. Andrada and Salamon [**4**] have shown that any para-hypercomplex structure on a real Lie algebra $\mathfrak{g}$ rise to a hypercomplex structure on its complexification $\mathfrak{g}^{\mathbf{C}}$ (considered as a real Lie algebra). They referred to para-hypercomplex structure as complex product structure.

An additional interest in integrable hypercomplex and para-hypercomplex structure is provided by the fact that each of these structures

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