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ON THE SPACE OF ORIENTED GEODESICS OF HYPERBOLIC 3-SPACE

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ABSTRACT. We construct a Kähler structure $(\mathbf{J}, \Omega, \mathbf{G})$ on the space $\mathbf{L}(\mathbf{H}^3)$ of oriented geodesics of hyperbolic 3-space $\mathbf{H}^3$ and investigate its properties. We prove that $(\mathbf{L}(\mathbf{H}^3), \mathbf{J})$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 - \overline{\Delta}$, where $\overline{\Delta}$ is the reflected diagonal, and that the Kähler metric $\mathbf{G}$ is of neutral signature, conformally flat and scalar flat. We establish that the identity component of the isometry group of the metric $\mathbf{G}$ on $\mathbf{L}(\mathbf{H}^3)$ is isomorphic to the identity component of the hyperbolic isometry group. Finally, we show that the geodesics of $\mathbf{G}$ correspond to ruled minimal surfaces in $\mathbf{H}^3$, which are totally geodesic if and only if the geodesics are null.

1. Introduction. The space $\mathbf{L}(\mathbf{M}^3)$ of oriented geodesics of a 3-manifold $\mathbf{M}^3$ of constant curvature is a 4-dimensional manifold which carries a natural complex structure $\mathbf{J}$. In the case where $\mathbf{M}^3$ is an Euclidean 3-space $\mathbf{E}^3$, this complex structure can be traced back to Weierstrass [**13**] and Whittaker [**14**], with its modern reemergence occurring in Hitchen's study of monopoles on $\mathbf{E}^3$ [**5**].

More recently, this structure has been supplemented by a compatible symplectic structure, so that $\mathbf{L}(\mathbf{M}^3)$ inherits a natural Kähler structure. This has been investigated when $\mathbf{M}^3 = \mathbf{E}^3$ and $\mathbf{M}^3 = \mathbf{E}^3_1$ [**2, 3, 4**], and the purpose of this paper is to study the hyperbolic 3-space case $\mathbf{M}^3 = \mathbf{H}^3$.

From a topological point of view, $\mathbf{L}(\mathbf{M}^3)$ is homeomorphic to $\mathrm{S}^2 \times \mathrm{S}^2 - \Delta$, where Δ is the diagonal. However, from holomorphic point of view we show that:

Theorem 1. *The complex surface* $(\mathbf{L}(\mathbf{H}^3), \mathbf{J})$ *is biholomorphic to* $\mathbf{P}^1 \times \mathbf{P}^1 - \overline{\Delta}$, *where* $\overline{\Delta}$ *is the reflected diagonal* (*see Definition* 2).

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