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THE RATLIFF-RUSH CLOSURE OF INITIAL IDEALS OF CERTAIN PRIME IDEALS

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ABSTRACT. Let K be a field, and let $m_0, \dots, m_n$ be an almost arithmetic sequence of positive integers. Let C be a monomial curve in the affine $(n+1)$-space, defined parametrically by $x_0 = t^{m_0}, \dots, x_n = t^{m_n}$. In this article we prove that the initial ideal of the defining ideal of C is Ratliff-Rush closed.

The Ratliff-Rush closure. Let R be a commutative Noetherian ring with unity and I a regular ideal in R, that is, an ideal that contains a nonzero divisor. Then the ideals of the form $I^{n+1} : I^n = \{x \in R \mid xI^n \subseteq I^{n+1}\}$ give the ascending chain $I : I^0 \subseteq I^2 : I^1 \subseteq \cdots \subseteq I^n : I^{n+1} \subseteq \cdots$. Let us denote

$$\widetilde{I} = \bigcup_{n \geq 1} (I^{n+1} : I^n).$$

As R is Noetherian, $\widetilde{I} = I^{n+1} : I^n$ for all sufficiently large n. Ratliff and Rush [**8**, Theorem 2.1] proved that $\widetilde{I}$ is the unique largest ideal for which $(\widetilde{I})^n = I^n$ for sufficiently large n. The ideal $\widetilde{I}$ is called the Ratliff-Rush *closure* of I, and I is called *Ratliff-Rush closed* if $I = \widetilde{I}$. It is easy to see that $I \subseteq \widetilde{I}$ and that an element of $(I^n : I^{n+1})$ is an integral over I. Hence, for all regular ideals I,

$$I \subseteq \widetilde{I} \subseteq \bar{I} \subseteq \sqrt{I},$$

where $\bar{I}$ is the integral closure of I. Thus, all radical and integrally closed regular ideals are Ratliff-Rush closed. But there are many ideals which are Ratliff-Rush closed but not integrally closed. For example, the ideal $I = (x^2, y^2) \subset k[x, y]$ is clearly not integrally closed as $xy \in \bar{I}$. Note that if $(xy)\, I^n \subseteq I^{n+1}$ for some n, then by the x-degree count we must have $(xy)\left(y^2\right)^n \in \left(y^2\right)^{n+1}$ which contradicts the y-degree count. Hence $xy \notin \widetilde{I}$. As $\widetilde{I} \subseteq \bar{I} = (x^2, xy, y^2)$, then I is Ratliff-Rush closed.

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