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ON A-CONVEX NORMS ON COMMUTATIVE ALGEBRAS

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ABSTRACT. We study such norms on commutative algebras for which the multiplication is separately continuous. By comparing a given norm $\| \ \|$ to its operator semi-norm $\| \ \|_{\rm op}$, we get two constants $m(\| \ \|)$ (the modulus of m-convexity) and $r(\| \ \|)$ (the modulus of regularity). We study how these constants are connected to the m-convexity and to the A-convexity of $\| \ \|$. In particular, we give a concept of an irregular norm and study some properties of such norms. Further, we will give a generalization of the famous theorem of Gelfand, which states that a complete A-convex norm $\| \ \|$ is always equivalent to some m-convex norm $| \ |$, and if the algebra has a unit element e, this norm can be chosen so that $|e| = 1$.

1. Introduction. In this paper, A will denote a commutative algebra over the field $\mathbf{C}$ of complex numbers. If A has a unit element, it will be denoted by e. Let $\| \ \|$ be a usual linear-space norm on A. The topology on A defined by $\| \ \|$ will be denoted by $T(\| \ \|)$. It is said that the multiplication on A is separately continuous with respect to the norm $\| \ \|$, if the mapping $(x, y) \mapsto xy$ from $A \times A$ into A is continuous with respect to one component, when the other one is fixed ($A \times A$ is provided with the usual product topology induced by $T(\| \ \|)$). Moreover, the multiplication on A is said to be jointly continuous with respect to the norm $\| \ \|$, if the mapping $(x, y) \mapsto xy$ from $A \times A$ into A is continuous with respect to both components at the same time. The norm $\| \ \|$ is said to be absorbingly convex (shortly A-convex) on A, if for each x in A there exists a constant $M_x \geq 0$ (depending on x) such that

$$\|xy\| \leq M_x \|y\| \quad \text{for all } y \text{ in } A.$$

Moreover, the norm $\| \ \|$ is said to be submultiplicative or multiplicatively convex (shortly m-convex) on A, if

$$\|xy\| \leq \|x\| \|y\| \quad \text{for all } x \text{ and } y \text{ in } A.$$

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