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EIGENVALUE PROBLEMS OF A DEGENERATE QUASILINEAR ELLIPTIC EQUATION

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ABSTRACT. This paper is concerned with positive eigenvalues and positive eigenfunctions of a class of degenerate and nondegenerate quasilinear elliptic equations. The degenerate property of the quasilinear operator can lead to a very different positive eigenvalue distribution when compared with classical linear eigenvalue problems.

1. Introduction. In the eigenvalue problem

$$-\nabla \cdot (D(\phi)\nabla\phi) = \lambda\phi \text{ in } \Omega, \quad \phi(x) = 0 \text{ on } \partial\Omega, \tag{1.0}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$; if $D(\phi) = D_0$ is a positive constant, then the problem has a countable number of eigenvalues and a positive eigenfunction only associated with the smallest eigenvalue. However, the eigenvalue distribution can be rather different if $D(\phi)$ depends on ϕ, especially in the degenerate case where $D(0) = 0$. In this note we investigate the eigenvalue problem for a slightly more general equation of the form

$$\begin{gathered} -\nabla \cdot (a(x)D(\phi)\nabla\phi) + \mathbf{c}(x) \cdot (D(\phi)\nabla\phi) = \lambda\phi \text{ in } \Omega \\ \phi(x) = 0 \text{ on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $a(x)$ is a strictly positive function in $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, $\mathbf{c}(x) = (c_1(x), \ldots, c_n(x))$ is a smooth function in Ω, and $D(\phi)$ is a positive function in $(0, \infty)$ with either $D(0) = 0$ or $D(0) > 0$. We assume that Ω is of class $C^{2+\alpha}$, $a(x)$ and $c_i(x)$, $i = 1, \ldots, n$, are in $C^{\alpha}(\overline{\Omega})$, and $D(\phi)$ satisfies hypothesis (H) in Section 2, where $\alpha \in (0, 1)$. Our aim is to show that, under the above condition, every $\lambda > 0$ is an eigenvalue of (1.1), and corresponding to it there is a positive eigenfunction $\phi(x)$.

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