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THE SMALLEST AMOUNT OF CHAOS

ALEJO BARRIO BLAYA AND VÍCTOR JIMÉNEZ LÓPEZ

ABSTRACT. In this paper we argue that the property that all nonapproximately periodic points form asymptotic pairs can be used to characterize those among chaotic dynamical systems having the least complicated behavior. Here, for a dynamical system (X, T, Φ), we say that a point $x \in X$ is *approximately periodic* if, for every $\varepsilon > 0$, there is a periodic point p such that $\limsup_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,p)) < \varepsilon$, and we say that points $x, y \in X$ form an *asymptotic pair* if $\lim_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,y)) = 0$. In support of this thesis we introduce a preorder in the class of properties of dynamical systems and show that, under certain conditions, this property is "smaller" than several other notions naturally related to the idea of chaotic behavior.

1. Introduction. Sometimes (see, e.g., [**5**, page 8]) a *dynamical system* (shortly, DS) is defined as a triplet (X, T, Φ) consisting of an additive semi-group T of $\mathbf{R}$ (the *set of times*), a set X (the *phase space*) and a map $\Phi : T \times X \to X$ (the *flow*) satisfying $\Phi(0, x) = x$ for every $x \in X$ and $\Phi(t, \Phi(s, x)) = \Phi(t+s, x)$ for every $t, s \in T$ and every $x \in X$. Depending on the point of view we are interested in, it is customary to establish some minimal requirements for both X and Φ. For instance, in the probabilistic setting X is a measure space and Φ is measurable. In this paper we adhere to the purely topological approach; hence, we assume that X is a topological space and Φ is continuous.

As it is, there is nothing wrong with the map Φ but one immediately sees that some additional restrictions must still be imposed both on T and X. For instance, since we aim to investigate the behavior of the *orbits* $\Phi_x(t) := \Phi(t, x)$ (we also use the word "orbit" to refer to the set $\Phi_x(T)$) of the system in the far future, T must accumulate at ∞, that is, sets of times like $\mathbf{Z}^- \cup \{0\}$ or $\mathbf{R}^- \cup \{0\}$ should not be considered. Besides, as a minimum technical requirement (see for instance Lemma 3.2), one

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