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NIKOL'SKII INEQUALITIES FOR LORENTZ SPACES

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ABSTRACT. A general approach is given for establishing Nikol'skii-type inequalities for various Lorentz spaces. The key ingredient for the proof is either a Bernstein-type inequality or a Remez-type inequality. Applications of our results to trigonometric polynomials on the torus T^d, algebraic polynomials on $[-1,1]$, spherical harmonic polynomials on the unit sphere S^{d-1} in $\mathbf{R}^d$, algebraic polynomials on $\mathbf{R}$ with Freud's weights and others will be presented.

1. Introduction. The Lorentz space $L_{p,q} \equiv L_{p,q}(\Omega,\mu)$, $0 < p, q \le \infty$, is the class of measurable functions on Ω with respect to the nonnegative measure μ satisfying $\|f\|_{p,q} < \infty$ where

$$(1.1)\quad \begin{aligned} \|f\|_{p,q} &:= \|f\|_{L_{p,q}(\Omega)} := \left\{ \frac{q}{p} \int_0^\infty t^{(q/p)-1} f^*(t)^q \, dt \right\}^{1/q}, \\ &\qquad 0 < p, \quad q < \infty, \\ \|f\|_{p,\infty} &:= \|f\|_{L_{p,\infty}(\Omega)} := \sup_{0<t<\mu(\Omega)} t^{1/p} f^*(t), \quad 0 < p \le \infty, \end{aligned}$$

and f^* is the nonincreasing rearrangement of f. Note that for $p = \infty$ the space $L_{\infty,q}$ is defined only for $q = \infty$.

We recall that the distribution function $\mu_f(\lambda)$ is given by

$$(1.2)\quad \mu_f(\lambda) = \mu(x \in \Omega : |f(x)| > \lambda),$$

and f^*, the rearrangement of f, is given by

$$(1.3)\quad f^*(t) = \inf(\lambda : \mu_f(\lambda) \le t).$$

For the classes of functions $\{\mathcal{N}_\nu\}_{\nu \in \mathbf{R}_+}$ for which $\mathcal{N}_\nu \subset L_{p,q}(\Omega,\mu)$ for $0 < p, q \le \infty$, a Nikol'skii-type inequality is

$$(1.4)\quad \|f\|_{p_2,q_2} \le C\Psi(\nu) \|f\|_{p_1,q_1} \quad \text{for } f \in \mathcal{N}_\nu.$$

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