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THETA FUNCTIONS ON THE THETA DIVISOR

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ABSTRACT. We show that the gradient and the Hessian of the Riemann theta function in dimension n can be combined to give a theta function of order $n+1$ and modular weight $(n+5)/2$ defined on the theta divisor. It can be seen that the zero locus of this theta function essentially gives the ramification locus of the Gaussian map. For Jacobians this leads to a description in terms of theta functions and their derivatives of the Weierstrass point locus on the associated Riemannian surface.

In the analytic theory of the Riemann theta function a natural place is taken by the study of the first and second order terms of its Taylor series expansion along the theta divisor. The first order term essentially gives the gradient, and hence the tangent bundle on the smooth locus, whereas the second order terms give rise to Hessians, which are widely recognized as carrying subtle geometric information along the singular locus of the theta divisor. For example, in the case of a Jacobian, these Hessians define quadrics containing the canonical image of the associated Riemann surface. Or, in the general case, one could investigate the properties of those principally polarized Abelian varieties that have a singular point of order two on their theta divisor, such that the Hessian of the theta function at that singular point has a certain given rank. This is a recent line of investigation begun by Grushevsky and Salvati Manni, with interesting connections to the Schottky problem [**6, 7**].

If one moves outside the singular locus of the theta divisor, it is not immediately clear whether the Hessian of the theta function continues to have some geometric significance. In this paper we prove that it does. More precisely, we show that a certain combination of the gradient and the Hessian gives rise to a well-defined theta function living on the theta divisor. We can compute its transformation behavior, i.e., its order and

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