ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 1, 2010

THE METHOD OF UPPER AND LOWER SOLUTIONS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS

MOUFFAK BENCHOHRA, SAMIRA HAMANI AND JUAN J. NIETO

ABSTRACT. In this paper, we prove the existence of solutions of second order differential inclusion with integral boundary conditions. We rely on the nonlinear alternative of Leray-Schauder combined with the lower and upper solutions method.

1. Introduction. This paper is concerned with the existence of solutions of second order differential inclusion with integral boundary conditions. We consider the following second order differential inclusion with integral boundary conditions:

$$x''(t) + \lambda x'(t) \in F(t, x(t)), \text{ almost everywhere } t \in [0,1], \tag{1}$$

$$x(0) = a, \tag{2}$$

$$x(1) = \int_0^1 g(x(s))\, ds, \tag{3}$$

where $F : [0,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a compact valued multi-valued map, $\mathcal{P}(\mathbf{R})$ is the family of all subsets of $\mathbf{R}$, $\lambda > 0$, $a \in \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ is continuous. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamic [**11**] and cellular systems [**1**]. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [**21, 22**], and the

2010 AMS *Mathematics subject classification.* Primary 34A60, 34B15.
Keywords and phrases. Differential inclusion, boundary value problem, integral boundary conditions, lower and upper solution, truncation map, fixed point.
The research of the third author was partially supported by Ministrio de Educacion y Cienca and FEDER, project MTM2004-06652-C03-01, and by Xunta de Galicia and FEDER, project PGIDIT05PXIC20702PN.
Received by the editors on November 22, 2006, and in revised form on September 25, 2007.

DOI:10.1216/RMJ-2010-40-1-13