ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

ON e-POWER b-HAPPY NUMBERS

XIA ZHOU AND TIANXIN CAI

ABSTRACT. Let e and b be positive integers; an e-power b-happy number is a positive integer, a such that $S^r_{e,\ b}(a) = 1$ for some $r \geq 0$. Here $S_{e,\ b}(a)$ is the sum of the eth powers of digits of a in base b and $S^r_{e,\ b}(a) = S_{e,\ b}(S^{r-1}_{e,\ b}(a))$. Let

$$\mathcal{A} = \{p \text{ prime} : p \mid (b-1) \text{ and } (p-1) \mid (e-1)\}, \quad P = \prod_{p \in \mathcal{A}} p.$$

In this paper, we prove that arbitrarily long sequences of P-consecutive e-power b-happy numbers exist for any e, b.

1. Introduction. For $a \in \mathbf{Z}^+$, we define $S_2(a)$ as the sum of the squares the decimal digits of a. For $a \in \mathbf{Z}^+$, let $S^0_2(a) = a$, and for $r \geq 1$, let $S^r_2(a) = S_2(S^{r-1}_2(a))$. A happy number is a positive integer a such that $S^r_2(a) = 1$ for some $r \geq 0$. In [**4**], Guy asked whether there exist sequences of consecutive happy numbers of arbitrary length. In 2000, El-Sedy and Siksek [**1**] gave an affirmative answer to this question.

Let e and b be positive integers. In 2001, Grundman and Teeple [**2**] first defined the so-called e-power b-happy number, i.e., they named positive integer a an e-power b-happy number if $S^r_{e,\ b}(a) = 1$ for some $r \geq 0$: here $S_{e,\ b}(a)$ is the sum of the eth powers of the digits of a in base b and $S^r_{e,\ b}(a) = S_{e,\ b}(S^{r-1}_{e,\ b}(a))$.

Let

$$\mathcal{A} = \{p \text{ prime} : p \mid (b-1) \text{ and } (p-1) \mid (e-1)\}, \qquad P = \prod_{p \in \mathcal{A}} p.$$

If $P > 1$ for some e and b, there are no consecutive e-power b-happy numbers. In fact, for any $p \in \mathcal{A}$,

$$S_{e,\ b}\bigg(\sum_{j=0}^{k} a_j \times b^j\bigg) \equiv \sum_{j=0}^{k} a_j^e \equiv \sum_{j=0}^{k} a_j \equiv \sum_{j=0}^{k} a_j \times b^j \pmod{p}.$$

2000 AMS *Mathematics subject classification.* Primary 11A07, 11A63.
Keywords and phrases. Happy numbers, cycles of numbers, digital problems.
This research was supported by the National Natural Science Foundation of China No. 10871169.
Received by the editors on November 9, 2006, and in revised form on June 13, 2007.
DOI:10.1216/RMJ-2009-39-6-2073