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THE HOGGATT-BERGUM CONJECTURE ON $D(-1)$-TRIPLES $\{F_{2k+1}, F_{2k+3}, F_{2k+5}\}$ AND INTEGER POINTS ON THE ATTACHED ELLIPTIC CURVES

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ABSTRACT. Denote by F_n the nth Fibonacci number. We show that if a positive integer d satisfies the property that for an integer $k \geq 0$ each of $F_{2k+1}d+1$, $F_{2k+3}d+1$ and $F_{2k+5}d+1$ is a perfect square, then d must be $4F_{2k+2}F_{2k+3}F_{2k+4}$. Using this result, we further show that if for an integer $k \geq 1$ the rank of the attached elliptic curve

$$E_k : y^2 = (F_{2k+1}x + 1)(F_{2k+3}x + 1)(F_{2k+5}x + 1)$$

over $\mathbf{Q}$ equals one, then the integer points on E_k are given by

$$(x, y) \in \{(0, \pm 1), (4F_{2k+2}F_{2k+3}F_{2k+4}, \pm(2F_{2k+2}F_{2k+3} + 1) \times (2F_{2k+3}^2 - 1)(2F_{2k+3}F_{2k+4} - 1))\}.$$

1. Introduction. Diophantus found that the rational numbers $1/16$, $33/16$, $68/16$, $105/16$ have the property that the product of any two of them increased by one is a square of a rational number. The first example of four positive integers with such a property was found by Fermat, which was the set $\{1, 3, 8, 120\}$. Replacing "one" by "n" leads to the following definition.

Definition 1. Let n be a nonzero integer. A set $\{a_1, \dots, a_m\}$ of m distinct positive integers is called a Diophantine m-tuple with the property $D(n)$ (or a $D(n)$-m-tuple) if $a_i a_j + n$ is a perfect square for all i, j with $1 \leq i < j \leq m$.

In case $n = 1$, a folklore conjecture says that a $D(1)$-quintuple does not exist. This is an immediate consequence of the following:

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