ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

CONVERSE JENSEN INEQUALITY

P.J. FITZSIMMONS

ABSTRACT. We use Skorokhod's embedding theorem to give a new proof of a converse to Jensen's inequality.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let X be an element of $L^1(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra, and define

$$Y := \mathbf{E}[X|\mathcal{G}],$$

an element of $L^1(\Omega, \mathcal{G}, \mathbf{P})$. If $\varphi : \mathbf{R} \to \mathbf{R}$ is convex, then the conditional form of Jensen's inequality asserts that

$$\mathbf{E}[\varphi(X)|\mathcal{G}] \geq \varphi(\mathbf{E}[X|\mathcal{G}]) = \varphi(Y), \quad \text{a.s.},$$

part of the assertion being that the expectations are almost surely well-defined. In particular,

$$\mathbf{E}[\varphi(X)] \geq \mathbf{E}[\varphi(Y)], \tag{1}$$

with an analogous stipulation. The following converse assertion seems to be well known, see [**2, 3**]. The proof we present may have some claim to novelty. We write $X \stackrel{\mathrm{d}}{=} Y$ to indicate that random variables X and Y have the same distribution.

Theorem. *Let X and Y be integrable random variables such that* (1) *holds for all convex φ. Then there is a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, a random variable $X' \in L^1(\Omega', \mathcal{F}', \mathbf{P}')$, and a σ-algebra $\mathcal{G}' \subset \mathcal{F}'$ such that $X \stackrel{\mathrm{d}}{=} X'$ and $Y \stackrel{\mathrm{d}}{=} \mathbf{E}[X' \mid \mathcal{G}']$.*

Proof. Taking $\varphi(x) = x$ and then $\varphi(x) = -x$, we see that $\mathbf{E}[X] = \mathbf{E}[Y]$. Let $B = (B_t)$ be a one-dimensional Brownian motion defined

2000 AMS *Mathematics subject classification.* Primary 60J65, Secondary 26A51, 26D15.
Keywords and phrases. Jensen's inequality, Skorokhod stopping, Brownian motion.
Received by the editors on June 18, 2007, and in revised form on July 17, 2007.
DOI:10.1216/RMJ-2009-39-6-1905