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CYCLIC COVERS OF RATIONAL ELLIPTIC SURFACES

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ABSTRACT. We compute the maximal rank of a cyclic cover for a class of rational elliptic surfaces.

1. Introduction. Let $\pi_1 : E_1 \to \mathbf{P}^1$ be a smooth complex relatively minimal nonisotrivial elliptic surface with section, and consider the map $\mathbf{P}^1 \to \mathbf{P}^1$ defined by $t \to t^r$. Define $\pi_r : E_r \to \mathbf{P}^1$ to be the minimal compactification of the Neron model of the generic fiber of $E \times_{\mathbf{P}^1} \mathbf{P}^1$.

For $t \in \mathbf{P}^1$, let E_1^t be the fiber of E_1 over t with conductor f_t and Euler characteristic e_t. If the fiber is of type I_n or I_n^*, let $n_t = n$ and set $n_t = 0$ otherwise. In [**2**] we give a bound for the rank of E_r if

$$\gamma = \sum_{t \neq 0, \infty} (f_t - e_t/6) - \frac{n_0 + n_\infty}{6} < 1.$$

However, this bound is far from sharp.

Persson lists all 287 possible configurations of singular fibers on a rational elliptic surface [**3**]. Thirty-eight of these have $\gamma < 1$. When $\gamma = 0$, either E_1 is semi-stable or all fibers are of type I_n or I_n^*. We have already shown [**2**] that in the nine cases where $\gamma = 0$, E_r is extremal for all r and thus E_r has rank 0 for all r.

In this paper, we consider the remaining 29 cases where E_1 is a rational elliptic surface with $0 < \gamma < 1$ and compute the rank of E_r in most of these and significantly improve the bound given in [**2**] in the rest.

We will see that our bounds depend only on the fibers at $t = 0$ and $t = \infty$. Because of this, our bounds hold for all (not necessarily rational) elliptic surfaces with $\gamma < 1$ and the given fiber types at 0 and ∞.

2. Preliminary results. Unless otherwise noted, proofs of the results in this section can be found in [**2**, Section 1]. For any elliptic surface $\pi : E \to \mathbf{P}^1$,

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