ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

DOUBLE EULER SUMS ON HURWITZ ZETA FUNCTIONS

MINKING EIE AND WEN-CHIN LIAW

ABSTRACT. In an attempt to derive explicit evaluation of the double Euler sum with Dirichlet characters defined by

$$S^{\chi}_{p,q} := \sum_{k=1}^{\infty} \frac{\chi(k)}{k^q} \sum_{j=1}^{k} \frac{1}{j^p},$$

we decompose it into double Euler sums on Hurwitz zeta functions. In this paper, we give explicit evaluation of these Euler sums on Hurwitz zeta functions in terms of Hurwitz zeta values and the digamma function.

1. Introduction. For a pair of positive integers p and q with $q \geq 2$, the classical (double) Euler sum is defined as

$$S_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}. \tag{1.1}$$

It is well known that $S_{p,q}$ can be evaluated in terms of values at positive integers of the Riemann zeta function when $p = 1$ or $(p,q) = (2,4)$ or $(p,q) = (4,2)$ or $p = q$ or $p+q$ is odd. For explicit evaluations of the classical Euler sums or related alternating sums, readers may consult [**2, 3, 6, 11, 16, 18**]. Note that our $S_{p,q}$ is referred to as the double Harmonic series and denoted by $S(q,p)$ in [**12, 13**] but is called double zeta-star value and denoted by $\zeta^*(q,p)$ in [**1**]. It now becomes more customary to denote the double zeta values by $\zeta(s_1, s_2)$, which unfortunately conflicts with equally standard $\zeta(s,x)$ notation for the

2000 AMS *Mathematics subject classification.* Primary 11M06, Secondary 11M35, 33B15.
Keywords and phrases. Euler sums, Hurwitz zeta function.
The first author was supported by the Department of Mathematics, National Chung Cheng University and by the National Science Council of Taiwan, Republic of China. The second author was partially supported by the National Science Council of Taiwan, Republic of China, through grant NSC94-2115-M-194-009.
Received by the editors on August 10, 2006, and in revised form on July 6, 2007.
DOI:10.1216/RMJ-2009-39-6-1869