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A NOTE ON ARONSSON'S EQUATION

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ABSTRACT. This note gives reasons why the equation in the title is interesting, shows that constant multiples of solutions of the eikonal equation are solutions, and proves that, for other local solutions, curves on which the length of the gradient remains constant propagate with a normal velocity depending on curvature and "time." Under reasonable assumptions, this conforms to the setting used by Souganidis in [**10**] to study propagating fronts.

1. Introduction. This note considers C^2 solutions f, in some open set $\Omega \subseteq \mathbf{R}^2$, of the equation

$$(f_x)^2 f_{xx} + 2 f_x f_y f_{xy} + (f_y)^2 f_{yy} = 0. \tag{1}$$

As shown in [**8**], equation (1) is the condition under which each curve $\sigma(t)$ of steepest descent on the graph of f is asymptotic, i.e., its acceleration vector $\sigma''(t)$ remains tangent to the graph of f at each point on the curve $\sigma(t)$. Indeed, if $\sigma(t) = (x(t), y(t), f(x(t), y(t)))$, then to say that σ is a steepest descent curve means that its horizontal velocity (x', y') is a negative multiple of (f_x, f_y), the gradient of f. Thus, the condition that σ be asymptotic, namely,

$$(x')^2 f_{xx} + 2x'y' f_{xy} + (y')^2 f_{yy} = 0,$$

is equivalent to (1). Alternatively, (1) can be viewed as the requirement that at each point $P \in \Omega$ the second derivative of the values of f along the line through P in the direction of $\nabla f(P)$ is 0 at P, since the condition

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