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ON THE SIZE OF SETS IN WHICH $xy + 4$ IS ALWAYS A SQUARE

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ABSTRACT. In this paper, we prove that there does not exist a set of 7 positive integers such that the product of any two of its distinct elements increased by 4 is a perfect square.

1. Introduction. Let n be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

The problem of finding such sets was first studied by Diophantus in the case $n = 1$. He found a set of four positive rationals with the above property:

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}.$$

However, the first $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$, was found by Fermat. Later Euler was able to add the fifth positive rational, 777480/8288641, to Fermat's set, see [**5**], [**6**, pages 103–104, 232]. Recently, Gibbs [**17**] found examples of sets of six positive rationals with the property of Diophantus. The conjecture is that there does not exist a $D(1)$-quintuple. In 1969, Baker and Davenport [**1**] proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Recently, Dujella, see [**11**], proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples. This implies that there does not exist a $D(4)$-8-tuple and that there are only finitely many $D(4)$-septuples, see [**15**]. In this paper we will improve this result.

In the case $n = 4$ the conjecture is that there does not exist a $D(4)$-quintuple. Actually there is a stronger version of that conjecture, see [**15**, Conjecture 1].

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