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A NOTE ON SOME CHARACTERIZATIONS OF ARITHMETIC FUNCTIONS

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An arithmetic function is a mapping from positive integers into the field of complex numbers. We shall denote the set of arithmetic functions by $\mathcal{A}$. Various binary product operations dependent on the divisibility properties of the natural number n may be defined on the set $\mathcal{A}$. One such well-known product is the Dirichlet convolution

$$(1) \qquad (f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right),$$

where $f, g \in \mathcal{A}$. A large number of analogues and generalizations of Dirichlet's convolution have been studied in the literature, and for further information, the reader is referred to [**3**–**7**]. In this paper, we investigate the functions defined by

$$(2) \qquad G_{(f\circ g)}(n,m) = \sum_{d|(n,m)} f(d) g\left(\frac{m}{d}\right)$$

and

$$(3) \qquad G_{(h\circ f\circ g)}(n,m) = \sum_{d|(n,m)} h(d) f\left(\frac{n}{d}\right) g\left(\frac{m}{d}\right)$$

where $f, g, h \in \mathcal{A}$, and n, m are natural numbers with (n,m) as the gcd of n and m. It follows that $G_{(f\circ g)}(m,m) = (f*g)(m)$. These functions play a role in the study of arithmetic functions within the context of Dirichlet convolution as will be demonstrated in this note.

Definitions. (i) An arithmetic function f which is not identically zero is called multiplicative if $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$,

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