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LIFTING PROPERTIES OF PRIME GEODESICS

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ABSTRACT. We continue the study begun by Sarnak and Stopple of prime geodesics on $\Gamma\backslash\mathcal{H}$, Γ the modular group. We now allow Γ to be a Fuchsian group whose matrix entries lie in the ring of integers $\mathcal{O}_K$ of a number field K. There is a one-to-one correspondence between the prime geodesics $\mathcal{P}$ on $\Gamma\backslash\mathcal{H}$ and the primitive hyperbolic conjugacy classes $\{\gamma\}$ in Γ. An eigenvalue ε of an element of $\{\gamma\}$ determines a quadratic extension field $K(\varepsilon)$ of K. On the other hand, a prime ideal Q of $\mathcal{O}_K$ determines covering surfaces of $\Gamma\backslash\mathcal{H}$. A Frobenius map relates the lifting of $\mathcal{P}$ to the splitting of Q in $K(\varepsilon)$.

1. Introduction. One of the most important objects of study in number theory and geometry is the *modular group*

$$\Gamma := SL\,(2,\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbf{Z} \text{ and } ad-bc=1 \right\},$$

which acts on the upper half-plane $\mathcal{H} := \{z \in \mathbf{C} : \mathrm{Im}\,(z) > 0\}$ via linear fractional transformation $z \mapsto (az+b)/(cz+d)$. The orbits of $\mathcal{H}$ under this action form a quotient surface which has *fundamental domain*

$$\Gamma\backslash\mathcal{H} := \{z \in \mathbf{C} : |z| > 1 \text{ and } |\mathrm{Re}\,(z)| < 1/2\}.$$

Gauss was the first to study the modular group when he explored the equivalence and reduction of binary quadratic forms. Gauss must have been aware of the interplay here between number theory and geometry: definite forms may be interpreted as points in $\mathcal{H}$, and indefinite forms may be interpreted as geodesic semicircles on $\mathcal{H}$. Reduction of forms is obtained by letting the modular group carry points or geodesics to the fundamental domain. This gives a geometric explanation for why the study of indefinite forms is more difficult than the study of definite forms.

If $\gamma \in \Gamma$ and $|\mathrm{tr}\,(\gamma)| > 2$, then γ is *hyperbolic* and determines two distinct real fixed points. The geodesic semi-circle on $\mathcal{H}$ joining

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