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GENERATING SINGULARITIES OF WEAK SOLUTIONS OF p-LAPLACE EQUATIONS ON FRACTAL SETS

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ABSTRACT. We study p-Laplace equations $-\Delta_p u = F(x)$ possessing weak solutions in the Sobolev space $W_0^{1,p}(\Omega)$, $\Omega \subset \mathbf{R}^N$, that are singular on prescribed fractal sets having large Hausdorff dimension. With an appropriate choice of $F \in L^{p'}(\Omega)$, the Hausdorff dimension of a singular set of the weak solution can be made arbitrarily close to $N - pp'$ if $pp' < N$. For $p = 2$, that is, for the classical Laplace equation, the bound $N-4$ is optimal, provided $N \geq 4$. Moreover, there exist maximally singular solutions, that is, such that the bound is achieved. The proof is obtained via an explicit lower a priori bound of supersolutions corresponding to special choice of righthand sides that are singular near a fractal set.

1. Introduction. Let Ω be an open set in $\mathbf{R}^N$ and $1 < p < \infty$. Throughout this paper we assume that $p < N$, so that functions from the Sobolev space $W^{1,p}(\Omega)$ may have discontinuities. It is well known that, for any function $F \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ is the conjugate exponent, there exists a unique weak solution u of the boundary value problem involving the p-Laplace equation:

$$-\Delta_p u = F(x) \quad \text{in } \mathcal{D}'(\Omega), \quad u \in W_0^{1,p}(\Omega). \tag{1}$$

We are interested in how large the Hausdorff dimension of the singular set of solutions of this equation can be, generated by righthand sides from $L^{p'}(\Omega)$. Let us recall the definition of the singular set $\operatorname{Sing} u$.

We say that $a \in \Omega$ is a singular point of a measurable function $u \colon \Omega \to \mathbf{R}$ if there exist positive constants γ, ε, C such that

$$u(x) \geq C \cdot |x-a|^{-\gamma} \quad \text{for almost every } x \in B_\varepsilon(a),$$

where $B_\varepsilon(a)$ is the open ball of radius ε around a.

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