ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 6, 2008

ON THE HEIGHTS OF HAPPY NUMBERS

TIANXIN CAI AND XIA ZHOU

ABSTRACT. A happy number is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$; here $S_2(a)$ is the sum of the squares of its decimal digits and $S_2^m(a) = S_2(S_2^{m-1}(a))$. The height of a happy number is the least $m \geq 0$ such that $S_2^m(a) = 1$. In this paper, we give a general method to find theoretically the least happy number of any given height. For instance, we determine the least happy numbers of heights 11 and 12.

1. Introduction. The function S_2 is defined so that for any $a \in Z^+$, $S_2(a)$ is the sum of the squares of its decimal digits. For $a \in Z^+$, let $S_2^0(a) = a$, and for $m \geq 1$, let $S_2^m(a) = S_2(S_2^{m-1}(a))$. A happy number is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$. It is well known [**5**] that 4 is not a happy number and that, in fact, for all $a \in Z^+$, a is not a happy number if and only if $S_2^m(a) = 4$ for some $m \geq 0$. The height of a happy number is the least $m \geq 0$ such that $S_2^m(a) = 1$. Hence, 1 is a happy number of height 0, 10 is a happy number of height 1, 13 is a happy number of height 2, 23 is a happy number of height 3, 19 is a happy number of height 4, and 7 is a happy number of height 5.

In [**4**] (problem E34), Guy asks several questions about happy numbers such as: do there exist sequences of consecutive happy numbers of arbitrary length? What is the bound of the least happy number with height h? In 2000, El-sedy and Siksek [**1**] gave an affirmative answer to the former question. For the latter question, by computing the heights of each happy number less than 400, it is easy to find the least happy numbers of heights up to 6. (These, as well as the least happy number of height 7, can be found in [**4, 6**].) In 1994, Guy [**4**] reports that McCrainie verified the value of the least happy number of height 7 and determined the value of the least happy number of height 8.

2000 AMS *Mathematics subject classification.* Primary 11A07, 11A63.
Keywords and phrases. Happy numbers, heights.
This research was supported by the National Natural Science Foundation of China No.10371107.
Received by the editors on January 12, 2006, and in revised form on April 19, 2006.
DOI:10.1216/RMJ-2008-38-6-1921