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A GENERALIZATION OF WOLSTENHOLME'S HARMONIC SERIES CONGRUENCE

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ABSTRACT. Let A, B be two nonzero integers. Define the Lucas sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by

$$u_0 = 0, \quad u_1 = 1, \quad u_n = Au_{n-1} - Bu_{n-2} \text{ for } n \geq 2$$

and

$$v_0 = 2, \quad v_1 = A, \quad v_n = Av_{n-1} - Bv_{n-2} \text{ for } n \geq 2.$$

For any $n \in \mathbf{Z}^+$, let w_n be the largest divisor of u_n prime to $u_1, u_2, \dots, u_{n-1}$. We prove that for any $n \geq 5$

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{w_n^2},$$

where $\Delta = A^2 - 4B$.

1. Introduction. Let A, B be two nonzero integers. Define the Lucas sequence $\{u_n\}_{n=0}^{\infty}$ by

$$u_0 = 0, \quad u_1 = 1 \quad \text{and} \quad u_n = Au_{n-1} - Bu_{n-2} \quad \text{for } n \geq 2.$$

Also its companion sequence $\{v_n\}_{n=0}^{\infty}$ is given by

$$v_0 = 2, \quad v_1 = A \quad \text{and} \quad v_n = Av_{n-1} - Bv_{n-2} \quad \text{for } n \geq 2.$$

Let $\Delta = A^2 - 4B$ be the discriminant of $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$. It is easy to show that

$$v_n = \alpha^n + \beta^n$$

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