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WHEN IS BETWEENNESS PRESERVED?

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ABSTRACT. Let V and W be vector spaces over an ordered field $\mathbf{F}$. A map $f : V \to W$ is said to preserve betweenness if, for every $v_1, v_2, v_3 \in V$, if v_2 is between v_1 and v_3 on a line, then $f(v_2)$ is between $f(v_1)$ and $f(v_3)$ on a line. Wetzel asks whether every injection $f : \mathbf{R}^2 \to \mathbf{R}^2$ which preserves betweenness must be affine. We give an affirmative answer after treating the problem in a more general setting.

1. Introduction. Let V and W be vector spaces over an ordered field $\mathbf{F}$ such that $\dim V \geq 2$. It is immediate that every affine injection f from V to W "preserves betweenness" in the sense that for any $v_1, v_2, v_3 \in V$, if v_2 is between v_1 and v_3 on a line, then $f(v_2)$ is between $f(v_1)$ and $f(v_3)$ on a line. But to what extent is the converse true? For this question, we will see that it suffices to assume $V = W = \mathbf{F}^2$, cf. Remark 1.2 below.

It is known that if $\mathbf{F} = \mathbf{R}$, and if $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a bijection that preserves betweenness, then f is affine (see e.g., [**1**, IV.10.4]). This uses the fact [**1**, IV.10.1] that any bijection of $\mathbf{R}^2$ which maps lines to lines is affine. Wetzel [**5**] raises a natural question:

Problem 1.1. *Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be an injection which preserves betweenness. Must f be affine?*

We will answer this question affirmatively (Corollary 4.6). The property of preserving nonbetweenness turns out to be critical. We will show that for any subfield $\mathbf{F} \subseteq \mathbf{R}$, if a map $f : \mathbf{F}^2 \to \mathbf{F}^2$ preserves betweenness and nonbetweenness, then f is affine.

We now define our terms.

Let V be a vector space over an ordered field $\mathbf{F}$. For $a, b \in V$, the closed segment and open segment between a and b, denoted by seg $[a, b]$

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