ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 4, Fall 1989

A LOCALIZATION OPERATOR FOR RATIONAL MODULES

JAMES LI-MING WANG

Let X be a compact subset of the complex plane $\mathbf{C}$ and let g be a continuous function on X. We denote by $\mathcal{R}(X,g)$ the rational module

$$\{r_0(z) + r_1(z)g(z)\},$$

where each r_i denotes a rational function with poles off X.

In the case that $g(z) = \overline{z}$, the closure of $\mathcal{R}(X,\overline{z})$ in various topologies was first considered by O'Farrell [**4**] and was applied to rational approximation problems in Lipschitz norm. Later, several authors (e.g., Carmona, Trent, Verdera and Wang) have gone into the subject. A question which arose from these investigations concerned the characterization of $R(X,g)$, the uniform closure of $\mathcal{R}(X,g)$ in $C(X)$ when X has empty interior $\dot{X}$. This was settled in [**5**] (also see [**1**]) by showing that $R(X,g) = C(X)$ if and only if $R(Z) = C(Z)$ where Z is the subset of X on which $\overline{\partial} g$ vanishes. Here $\overline{\partial}$ is the usual Cauchy-Riemann operator in the complex plane.

The existence of interior points, however, makes the problem more difficult. It is natural to ask the following question: Is

$$R(X,g) = \{f \in C(X) : \overline{\partial}(\overline{\partial} f/\overline{\partial} g) = 0 \text{ in } \dot{X}\}$$

whenever $\overline{\partial} g \neq 0$ on an arbitrary compact set X? In particular, when $g(z) = \overline{z}$, this should be viewed as the Mergelyan approximation problem for the operator $\overline{\partial}^2 = \overline{\partial} \circ \overline{\partial}$:

$$(*) \qquad \text{Is } R(X,\overline{z}) = \{f \in C(X) : \overline{\partial}^2 f = 0 \text{ in } \dot{X}\}$$

for an arbitrary compact set X?

For the case when X is a compact set whose complement is connected, the approximation problem is not too difficult. In [**1**], a standard

Received by the editors on November 10, 1986 and in revised form on February 12, 1987.