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ON ORDERINGS, VALUATIONS AND 0-PRIMES OF A COMMUTATIVE RING

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1. Introduction. In this paper we examine the interplay between the orderings and valuations of a commutative ring. In the field case, this well-known area of study, which has influenced work in both quadratic forms and real geometry, is developed quite nicely in several sources. Among these we make special note of the recent monographs of Lam [**5**] and Prestel [**7**], while, as an overview of semi-real rings and orderings, [**4**] serves well. We will also consider the 0-primes of a commutative ring, which were introduced in [**8**], and their relationship to orderings.

As we will see, our investigation is somewhat encumbered by the fact that there are two notions of a valuation of a commutative ring; the first can be found in Bourbaki [**1**] and the second is a refinement due to Manis [**6**]. It is not clear to the author which choice (if any) should be made; both have their advantages as well as shortcomings (although the latter is most preferred in the literature).

Throughout this paper, R will denote a commutative ring with unity (assuming all of the standard conventions). We will employ the following notation: $\sigma(R)$ denotes the sums of squares of $R; X(R)$, the space of all orderings of R and $\mathrm{RSpec}\,(R) := \{P \cap -P | P \in X(R)\}$. Also, $\mathbf{Q}^+$ (respectively $\mathbf{R}^+$) will refer to the set of all non-negative rational (respectively real) numbers, and, for A and B arbitrary sets, $A\backslash B := \{a \in A | a \notin B\}$. Lastly, given a totally ordered abelian group G (which we will always consider additively), we set $G^* = G \cup \{\infty\}$ while adopting the conventions that ∞ is larger than any element of G and $g + \infty = \infty + g = \infty$ for all $g \in G$.

2. Valuations.

DEFINITION. By a *valuation* of a commutative ring we will mean a map $v : R \to G^*$, where G is a totally ordered abelian group (possibly trivial), satisfying