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LEVELS OF QUATERNION ALGEBRAS

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The level of a ring with identity is the least integer n for which -1 is expressible as a sum of n squares. Pfister [**3**] proved that the level of a field must be a power of two, and later Dai, Lam and Peng [**2**] showed that any positive integer may occur as the level of a commutative ring. There seems to be nothing in the literature about this problem in the non-commutative case. In [**5**] a different notion of level is discussed involving the expression of -1 as a sum of products of squares. In this note we examine the usual notion of level for quaternion division algebras. We show that any power of two may occur and also that $2^k + 1$ occurs for all $k \geq 1$. We have no information on whether other integer values can occur as the level of a quaternion division algebra.

Let D be a quaternion division algebra.

DEFINITION. The *level* $s(D)$ is the least integer n such that $-1 = \sum_{i=1}^{n} x_i^2$ with each $x_i \in D$. If -1 is not expressible as a sum of squares then we define $s(D)$ to be infinity.

We write $D = \left(\frac{a,b}{F}\right)$, F a field, characteristic $\neq 2, i^2 = a, j^2 = b, ij = -ji$, etc.

We write T_D for the four-dimensional quadratic form $\langle 1, a, b, -ab\rangle$ and T_P for the three-dimensional form $\langle a, b, -ab\rangle$. (Note that, apart from a scalar factor 2, T_D is the usual trace form of D over F, i.e., the map $D \to F$, $x \to \mathrm{tr}(x), tr$ denoting the reduced trace.) We use the notation of [**4**] for quadratic forms.

LEMMA 1. *$s(D) \leq 2$ if and only if either $\langle 1, 1\rangle \perp T_D$ is isotropic or $\langle 1\rangle \perp 2 \times T_P$ is isotropic.*

PROOF. If $x \in D$, then $x = p + qi + rj + sk$ for p, q, r, s in F and so $x^2 = p^2 + aq^2 + br^2 - abs^2 + 2pqi + 2prj + 2psk$. Now $s(D) \leq 2$ implies