ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 2, Spring 1989

MORSE THEORY WITHOUT CRITICAL POINTS

WOLFGANG SMITH

Let X denote an n-dimensional differentiable manifold and $f : X \to R$ a real-valued differentiable function on X, where "differentiable" means (let us say) C^∞. We shall be concerned with the case where f has no critical points, and thus, too, with the case where X is not compact. In place of critical points we will introduce a notion of "critical fibers", and in place of the index we shall assign to each isolated critical fiber a set of "type numbers" m_p^+ for $p = 0, 1, \ldots, n-1$. Roughly speaking, a critical fiber is one across which the fiber-structure of f suffers a discontinuity, and m_p^+ is a homology measure of that discontinuity on dimension p. Given that f is bounded and has only a finite number of critical fibers, we let M_p^+ denote the sum of the type numbers m_p^+ over all critical fibers. Our main result is that these coefficients satisfy the strong Morse inequalities:

$$M_0^+ \geq R_0$$

$$M_1^+ - M_0^+ \geq R_1 - R_0$$

$$M_{n-1}^+ - M_{n-2}^+ + \cdots \pm M_0^+ = R_{n-1} - R_{n-2} + \cdots \pm R_0,$$

where R_p denotes the p-dimensional Betti number of X (with respect to a given coefficient module G). We show, moreover, that for $p < n-1$ they constitute in fact a *bona fide* generalization of the classical Morse inequalities. For, if $h : M \to R$ denotes a differentiable function on a compact manifold with non-degenerate critical points, and we let X denote the complement of the critical points in M, then our preceding inequalities for $f = h|X$ reduce (as will be shown) to the Morse inequalities for h on dimensions $p < n-1$.

1. Basic lemmas. First some notation and terminology. The symbol $H_p(X, A)$ will denote the p-dimensional singular homology group of the topological pair (X, A) over some (fixed) coefficient group G. We will say that the pair (X, A) is *regular* if the inclusion $A \subset X$ induces isomorphisms $H_p(A) \approx H_p(X)$ for all p. We will need the following elementary fact regarding excisive couples **[5]**: