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ON NORM AND ZERO ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS FOR GENERAL MEASURES, I

MATTHEW F. WYNEKEN

ABSTRACT. Let μ be a positive unit Borel measure with infinite support in the interval $I = [-1, 1]$, and let $\{p_n(x)\}_{n\geq 0}$ be the monic orthogonal polynomials associated to μ. Let ν_n denote the unit measure having mass $1/n$ at each zero of $p_n(x)$, and let $\lambda_n = (\int (p_n(x))^2 d\mu)^{1/2n}$. It is known that, subject to the condition that no set of unit μ-measure has capacity zero, when a subsequence of the sequence $\{\nu_n\}_{n\geq 1}$ converges, the corresponding subsequence of $\{\lambda_n\}_{n\geq 1}$ converges as well. We give an example which shows that the converse statement is not true in general.

Introduction and Definitions. We say that μ is a *weight measure* if it is a positive unit Borel measure with infinite support in the interval $I = [-1, 1]$, where the *support* $S(\mu)$ is the smallest closed set of unit μ-measure. Given a weight measure μ, for $n = 0, 1, 2, \ldots$, let $p_n(x) = x^n + \cdots$, denote the n^{th} orthogonal polynomial associated to μ, so that

$$\left(\int p_m(x) p_n(x) d\mu \right)^{1/2} = N_n \delta_{m,n},$$

where $\delta_{m,n} = \{0 \text{ if } m \neq n, \ 1 \text{ if } m = n\}$. Let ν_n denote the *zero measure* for $p_n(x)$, i.e., the unit measure with mass $1/n$ at each zero of $p_n(x)$, and let $\lambda_n = (N_n)^{1/n}$ be the n^{th} *linearized norm.* It is known that, subject to the condition that no set of unit μ-measure has capacity zero, when a subsequence of the sequence $\{\nu_n\}_{n\geq 1}$ converges, the corresponding subsequence of $\{\lambda_n\}_{n\geq 1}$ converges as well. Here we give an example which shows that the converse statement is not true in general.

If, for a unit Borel measure μ defined on I, we have $\lim_{n\to\infty} \int f d\nu_n = \int f d\mu$ for all functions f continuous on I,then we say that the sequence $\{\nu_n\}_{n\geq 1}$ *converges weakly* to μ or that μ is a *weak limit* of $\{\nu_n\}_{n\geq 1}$, and we write $\lim_{n\to\infty} \nu_n = \mu$. By a theorem of Helly, $\{\nu_n\}_{n\geq 1}$ always has weakly convergent subsequences.

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