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ON IKEBE'S CRITERION

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ABSTRACT. A 0-2 law for the metric projection is shown to hold in most of the common Banach spaces.

Let V be a linear subspace of the normed space E. Denote by P_V the (set-valued) metric projection of E onto V, $P_V x =: \{v \in V : ||x - v|| = d(x, V)\}$. V is called *proximinal* if $P_V x \neq \emptyset\ \forall x \in E$, *semichebyshev* if $|P_V x| \leq 1\ \forall x \in E$, and *Chebyshev* if both, i.e., if $|P_V x| = 1\ \forall x \in E$. If $v \in P_V x$, then $||x - v|| \leq ||x - 0|| = ||x||$, hence $||v|| \leq 2||x||$. For equality to hold, it is necessary that $||x - v|| = ||x||$, i.e., that $0 \in P_V x$. If V is semichebyshev, this implies that $v = 0$, hence $x = 0$. In [**8**], Ikebe showed that if V is a non-Chebyshev finite-dimensional subspace of $E = C[a, b]$, then there are $x \neq 0$ in E and $v \in P_V x$ with $||v|| = 2||x||$, so that

$$(*) \qquad ||v|| < 2||x|| \quad \forall x \in E,\ v \in P_V x$$

characterizes the Chebyshev property in this case.

Ikebe's proof uses the well-known Haar characterization of finite-dimensional Chebyshev subspaces of $C[a, b]$. In Singer's survey [**14**: Proposition 3.2, p. 28] it is observed that Ikebe's result holds also when $E = C(Q)$, Q *any* compact Hausdorff space. In the "added in proof" part of his survey (p. 92), Singer mentions a generalization to $E = C(Q, H)$, H a Hilbert space, due to K.H. Hoffmann [**7**].

Motivated by these results, we say that the normed space E has *Ikebe's property* (Ik) if, in E, every linear subspace satisfying (*) is semichebyshev. We say also that E has (Ik_1) (respectively, (Ik^1)) if this criterion is valid for all 1-dimensional (respectively, 1-codimensional) subspaces. Strictly convex spaces have the (Ik) trivially.

Geometrically, (Ik) (respectively (Ik_1) or (Ik^1)) means that, for every plane (respectively, line or hyperplane) F which supports the unit ball B_E at more than one point, there is a translate of F which supports

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